Int J Theor Phys (2007) 46: 3169–3181
DOI 10.1007/s10773-007-9432-0

Dynamic Entanglement and Separability Criteria for Quantum Computing Bit States

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Published online: 12 June 2007

Abstract A theoretical framework is demonstrated to evaluate the degree of entanglement of bit states in quantum computing. Separability of general superposition of Hilbert space unit vectors is discussed, and criteria in amplitude as well as in phase are derived. By these criteria the possibility of different quantum gates such as controlled not (CN), controlled controlled not (CCN), controlled rotation (CR), and controlled phase shift (CPS), to create the entanglement is examined. Furthermore, the selection of measurement mode external to the quantum system is incorporated in the formula using Kronecker delta (δ_{kx}), introducing the concept of *dynamic entanglement*. With this the process of wavefunction collapse upon measurement is understood as the result of the activation of the dynamic entanglement. A firefly in a box model is used to show a pure state of ontological uncertainty, which is in a dynamically entangled state in Hilbert space.

Keywords Dynamic entanglement · Separability · Quantum computing

1 Introduction

We have been investigating the mathematical measure and criteria for quantum entanglements,[1] deriving a graphical method to visualize the quantum computing process that changes the ontological uncertainty intrinsic for the configuration of the quantum bits [1, 2].

In this paper we discuss separability of superpositions of Hilbert space unit vectors, and derive criteria for separability using amplitude and phase. Using a firefly in a box model [3],

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[1] The entanglement is a superposition of product states, which can not be expressed as a simple product state. Originated from the German word Verschränkung used by Erwin Schrödinger.

we give an example of a state that violates the separability criteria, hence can be characterized as an entangled state. We also show that entangled states can be produced by quantum gates, such as controlled controlled not (CCN) and controlled rotation (CR). Further, we use Kronecker delta (δ_{kx}) to introduce "*dynamic entanglement*" (also understood as operational superposition) in reference to the selection of measurement mode external to the quantum system. Then the process of wavefunction collapse upon a measurement from outside is understood as the result of the activation of the dynamic entanglement.

2 Separability of Quantum State

2.1 3-bit State Separability

The status (ground or excited) of each bit in a 3-bit state is denoted in the ket $|\rangle$ (or bra $\langle|$) having the bit number as a subscript. As examples, $|0\rangle_i$ and $|1\rangle_i$ denote that bit i is in state 0 (ground state), or in state 1 (excited state), respectively. Further abbreviation is indicated below.

$$\begin{aligned}
&|0\rangle_1|0\rangle_2|0\rangle_3 \Rightarrow |000\rangle \Rightarrow |0\rangle, && |1\rangle_1|0\rangle_2|0\rangle_3 \Rightarrow |100\rangle \Rightarrow |4\rangle,\\
&|0\rangle_1|0\rangle_2|1\rangle_3 \Rightarrow |001\rangle \Rightarrow |1\rangle, && |1\rangle_1|0\rangle_2|1\rangle_3 \Rightarrow |101\rangle \Rightarrow |5\rangle,\\
&|0\rangle_1|1\rangle_2|0\rangle_3 \Rightarrow |010\rangle \Rightarrow |2\rangle, && |1\rangle_1|1\rangle_2|0\rangle_3 \Rightarrow |110\rangle \Rightarrow |6\rangle,\\
&|0\rangle_1|1\rangle_2|1\rangle_3 \Rightarrow |011\rangle \Rightarrow |3\rangle, && |1\rangle_1|1\rangle_2|1\rangle_3 \Rightarrow |111\rangle \Rightarrow |7\rangle.
\end{aligned}$$

A 3-bit state is said to be totally or perfectly separable when each bit can be factored out, and be in an intra-bit superposition to any extent [4]. Let us start from a perfectly separated 3-bit state Ψ_{123}, and expand it;

$$\begin{aligned}
\Psi_{123} &= (a_0e^{i\alpha_0}|0\rangle_1 + a_1e^{i\alpha_1}|1\rangle_1) \otimes (b_0e^{i\beta_0}|0\rangle_2 + b_1e^{i\beta_1}|1\rangle_2)\\
&\quad\otimes (c_0e^{i\gamma_0}|0\rangle_3 + c_1e^{i\gamma_1}|1\rangle_3)\\
&= a_0e^{i\alpha_0}b_0e^{i\beta_0}c_0e^{i\gamma_0}|0\rangle_1|0\rangle_2|0\rangle_3 + a_0e^{i\alpha_0}b_0e^{i\beta_0}c_1e^{i\gamma_1}|0\rangle_1|0\rangle_2|1\rangle_3\\
&\quad+ a_0e^{i\alpha_0}b_1e^{i\beta_1}c_0e^{i\gamma_0}|0\rangle_1|1\rangle_2|0\rangle_3 + a_0e^{i\alpha_0}b_1e^{i\beta_1}c_1e^{i\gamma_1}|0\rangle_1|1\rangle_2|1\rangle_3\\
&\quad+ a_1e^{i\alpha_1}b_0e^{i\beta_0}c_0e^{i\gamma_0}|1\rangle_1|0\rangle_2|0\rangle_3 + a_1e^{i\alpha_1}b_0e^{i\beta_0}c_1e^{i\gamma_1}|1\rangle_1|0\rangle_2|1\rangle_3\\
&\quad+ a_1e^{i\alpha_1}b_1e^{i\beta_1}c_0e^{i\gamma_0}|1\rangle_1|1\rangle_2|0\rangle_3 + a_1e^{i\alpha_1}b_1e^{i\beta_1}c_1e^{i\gamma_1}|1\rangle_1|1\rangle_2|1\rangle_3 \qquad (1)\\
&= A\,e^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + De^{i\lambda_D}|3\rangle\\
&\quad+ Ee^{i\lambda_E}|4\rangle + Fe^{i\lambda_F}|5\rangle + Ge^{i\lambda_G}|6\rangle + He^{i\lambda_H}|7\rangle. \qquad (2)
\end{aligned}$$

We have simplified notation for amplitudes and phases using:

$$A = a_0b_0c_0, \qquad \lambda_A = \alpha_0 + \beta_0 + \gamma_0, \tag{3}$$

$$B = a_0b_0c_1, \qquad \lambda_B = \alpha_0 + \beta_0 + \gamma_1, \tag{4}$$

$$C = a_0b_1c_0, \qquad \lambda_C = \alpha_0 + \beta_1 + \gamma_0, \tag{5}$$

$$D = a_0b_1c_1, \qquad \lambda_D = \alpha_0 + \beta_1 + \gamma_1, \tag{6}$$

$$E = a_1b_0c_0, \qquad \lambda_E = \alpha_1 + \beta_0 + \gamma_0, \tag{7}$$

$$F = a_1 b_0 c_1, \qquad \lambda_F = \alpha_1 + \beta_0 + \gamma_1, \tag{8}$$

$$G = a_1 b_1 c_0, \qquad \lambda_G = \alpha_1 + \beta_1 + \gamma_0, \tag{9}$$

$$H = a_1 b_1 c_1, \qquad \lambda_H = \alpha_1 + \beta_1 + \gamma_1. \tag{10}$$

Note that Ψ_{123} must satisfy the normalization condition

$$A^2 + B^2 + C^2 + D^2 + E^2 + F^2 + G^2 + H^2 = 1. \tag{11}$$

The amplitude condition and the phase condition for separability are now derived as follows. Ratios between amplitudes lead to the *amplitude condition*

$$\frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \frac{G}{H} = \frac{c_0}{c_1} \tag{12}$$

or alternatively

$$AD = BC, \qquad AF = BE, \qquad AH = BG, \qquad CF = DE. \tag{13}$$

The *phase condition* is

$$\alpha_0 + \alpha_1 + \beta_0 + \beta_1 + \gamma_0 + \gamma_1 = \lambda_A + \lambda_H = \lambda_B + \lambda_G = \lambda_C + \lambda_F = \lambda_D + \lambda_E \tag{14}$$

within trigonometric periodicity, or

$$\lambda_A - \lambda_B = \lambda_C - \lambda_D = \lambda_E - \lambda_F = \lambda_G - \lambda_H = \gamma_0 - \gamma_1. \tag{15}$$

The proof showing that if these conditions are fulfilled the original totally separated state is recovered, is given for the simpler 2-bit case.[2]

2.2 n-bit State Separability

In the same way as the 3-bit case, the n-bit totally separable state $\Psi_{12\cdots n}$ may be given as;

$$\begin{aligned}\Psi_{12\cdots n} = {} & (a_0 e^{i\alpha_{00}}|0\rangle_1 + a_1 e^{i\alpha_{01}}|1\rangle_1) \otimes (b_0 e^{i\alpha_{10}}|0\rangle_2 + b_1 e^{i\alpha_{11}}|1\rangle_2) \otimes \cdots \\ & \otimes (m_0 e^{i\alpha_{(n-2)0}}|0\rangle_{n-1} + m_1 e^{i\alpha_{(n-2)1}}|1\rangle_{n-1}) \otimes (n_0 e^{i\alpha_{(n-1)0}}|0\rangle_n + n_1 e^{i\alpha_{(n-1)1}}|1\rangle_n)\end{aligned}$$

[2] Given a general 2 bit system and the amplitude condition $k = \frac{A}{B} = \frac{C}{D}$ and phase condition $\phi = \lambda_A - \lambda_B = \lambda_C - \lambda_D$,

$$\begin{aligned}& A\, e^{i\lambda_A}|0\rangle + B e^{i\lambda_B}|1\rangle + C e^{i\lambda_C}|2\rangle + D e^{i\lambda_D}|3\rangle \\ & = kB e^{i(\lambda_B+\phi)}|0\rangle_1|0\rangle_2 + B e^{i\lambda_B}|0\rangle_1|1\rangle_2 + kD e^{i(\lambda_D+\phi)}|1\rangle_1|0\rangle_2 + D e^{i\lambda_D}|1\rangle_1|1\rangle_2 \\ & = B e^{i\lambda_B}|0\rangle_1[k e^{i\phi}|0\rangle_2 + |1\rangle_2] + D e^{i\lambda_D}|1\rangle_1[k e^{i\phi}|0\rangle_2 + |1\rangle_2] \\ & = [B e^{i\lambda_B}|0\rangle_1 + D e^{i\lambda_D}|1\rangle_1] \otimes \left[\frac{C}{D} e^{i(\lambda_C - \lambda_D)}|0\rangle_2 + |1\rangle_2\right] \\ & = [a_0 b_1 e^{i(\alpha_0+\beta_1)}|0\rangle_1 + a_1 b_1 e^{i(\alpha_1+\beta_1)}|1\rangle_1] \frac{e^{-i\beta_1}}{b_1}[b_0 e^{i\beta_0}|0\rangle_2 + b_1 e^{i\beta_1}|1\rangle_2] \\ & = [a_0 e^{i\alpha_0}|0\rangle_1 + a_1 e^{i\alpha_1}|1\rangle_1] \otimes [b_0 e^{i\beta_0}|0\rangle_2 + b_1 e^{i\beta_1}|1\rangle_2],\end{aligned}$$

thus the totally separated state is recovered.

$$= A_0 e^{i\lambda_{A_0}}|0\rangle + A_1 e^{i\lambda_{A_1}}|1\rangle + A_2 e^{i\lambda_{A_2}}|2\rangle + A_3 e^{i\lambda_{A_3}}|3\rangle + \cdots$$
$$+ A_{2^n-3} e^{i\lambda_{A_{2^n-3}}}|2^n-3\rangle + A_{2^n-2} e^{i\lambda_{A_{2^n-2}}}|2^n-2\rangle + A_{2^n-1} e^{i\lambda_{A_{2^n-1}}}|2^n-1\rangle \quad (16)$$

with the normalization conditions;

$$A_0^2 + A_1^2 + A_2^2 + A_3^2 + \cdots + A_{2^n-4}^2 + A_{2^n-3}^2 + A_{2^n-2}^2 + A_{2^n-1}^2 = 1. \quad (17)$$

All the coefficients $A_0, A_1, A_2, \ldots$ and phases $\lambda_{A_0}, \lambda_{A_1}, \lambda_{A_2}, \ldots$ have the same relations as indicated for the 3-bit case. Ratios between the first and the second terms, between the third and the fourth terms, and so on in (16) lead to the amplitude condition

$$\frac{A_0}{A_1} = \frac{A_2}{A_3} = \cdots = \frac{A_{2^n-4}}{A_{2^n-3}} = \frac{A_{2^n-2}}{A_{2^n-1}} = \frac{n_0}{n_1} \quad (18)$$

or alternatively

$$\begin{aligned} &A_0A_3 = A_1A_2, \qquad A_0A_5 = A_1A_4, \qquad A_0A_7 = A_1A_6, \qquad \ldots, \\ &A_{2^n-4}A_{2^n-1} = A_{2^n-2}A_{2^n-3}, \qquad \ldots \end{aligned} \quad (19)$$

as well as phase condition

$$\begin{aligned} \lambda_{A_0} - \lambda_{A_1} &= \lambda_{A_2} - \lambda_{A_3} = \cdots = \lambda_{A_{2^n-4}} - \lambda_{A_{2^n-3}} = \lambda_{A_{2^n-2}} - \lambda_{A_{2^n-1}} \\ &= \alpha_{(n-1)0} - \alpha_{(n-1)1} \end{aligned} \quad (20)$$

within trigonometric periodicity.

3 Dynamic Description of Entanglements

3.1 Firefly Model

As an interesting illustration of entanglement consider a three chamber firefly model [3]. This is a model with a three chamber triangular box having three sides (faces), on each of which is a window divided with a line vertically down its center, as shown in Fig. 1. The inside of the box is separated into three chambers, each one at a corner of the triangle. There is a path way or a slit on each wall between the chambers, so that a firefly can freely move into any of the chambers from any other chamber through the slits. There is one and only one firefly living in the box. It cannot occupy more than one chamber at a time, and lights or does not light of its own accord.

The physical experiment associated with this system is to look at one side of the box and record whether you see a light on the left half of the window, or on the right, or no light at all. Interestingly enough, this physical system behaves exactly like a quantum mechanical wave function (See Chap. 3 in [3]).

The state space of the system can be represented by a three-bit quantum register shown in Fig. 2, as

$$|\chi_1\rangle_1|\chi_2\rangle_2|\chi_3\rangle_3 = |\chi_1\chi_2\chi_3\rangle \quad (21)$$

where χ's represent the bit state 0 or 1, i.e. lit or not lit in windows looking into chambers 1, 2, or 3. The overall quantum state of the register is formulated as the superpositions and entanglements of the constituent bit states.

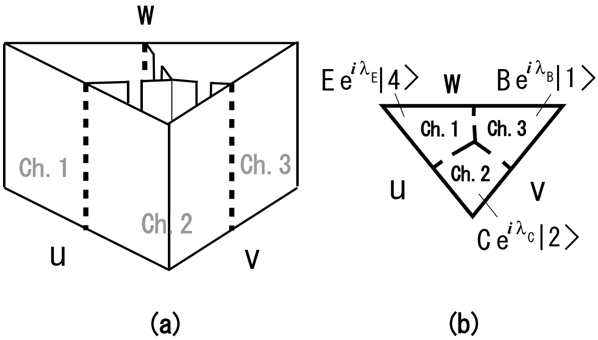

Fig. 1 The perspective view and the plan view of the three-chamber firefly box with the chambers, ch.1, ch.2, and ch.3, corresponding to the quantum states $|4\rangle$, $|2\rangle$, and $|1\rangle$ respectively. There is one and only one firefly in the box and it cannot occupy more than one chamber at a time. It can freely move into any chamber, and it can be lit or not lit of its own accord

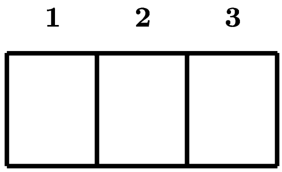

Fig. 2 Three-bit register for the three-chamber firefly system. The bits 1, 2, and 3 correspond to the chambers 1, 2, and 3 respectively, and also to the quantum states $|4\rangle$, $|2\rangle$, and $|1\rangle$ respectively

We can consider the three-chamber firefly system as represented by a three bit quantum register, and recall the general separable state given by (1) and (2) as

$$\begin{aligned}\Psi_{123} &= Ae^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + De^{i\lambda_D}|3\rangle \\ &\quad + Ee^{i\lambda_E}|4\rangle + Fe^{i\lambda_F}|5\rangle + Ge^{i\lambda_G}|6\rangle + He^{i\lambda_H}|7\rangle. \end{aligned} \tag{22}$$

However, since there is only one firefly, the product states having 1's in more than one bit are not allowed. For example, terms containing $|1\rangle_2\ |1\rangle_3$ must be eliminated. This leads to the conditions

$$A^2 + B^2 + C^2 + E^2 = 1$$

and

$$D = F = G = H = 0. \tag{23}$$

Then because $D = 0$, it is easily seen form (13) that

$$BC = AD = 0, \tag{24}$$

so that at least either $B = 0$ or $C = 0$. Thus, states for which $B \neq 0$ and $C \neq 0$ (such as $\frac{1}{\sqrt{3}}$ $|1\rangle + \frac{1}{\sqrt{3}}\ |2\rangle + \frac{1}{\sqrt{3}}\ |4\rangle$ i.e. "firefly lit") must be nonseparable and entangled.

As a matter of fact, (22) and (23) give a necessarily entangled quantum mechanical state Ψ_{3f}

$$\Psi_{3f} = A\ e^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + Ee^{i\lambda_E}|4\rangle \tag{25}$$

$$\begin{aligned} &= a_0e^{i\alpha_0}b_0e^{i\beta_0}c_0e^{i\gamma_0}|0\rangle_1|0\rangle_2|0\rangle_3 + a_0e^{i\alpha_0}b_0e^{i\beta_0}c_1e^{i\gamma_1}|0\rangle_1|0\rangle_2|1\rangle_3 \\ &\quad + a_0e^{i\alpha_0}b_1e^{i\beta_1}c_0e^{i\gamma_0}|0\rangle_1|1\rangle_2|0\rangle_3 + a_1e^{i\alpha_1}b_0e^{i\beta_0}c_0e^{i\gamma_0}|1\rangle_1|0\rangle_2|0\rangle_3. \end{aligned} \tag{26}$$

This state can be understood from three different points corresponding to three chambers by rearranging the terms. As an example for chamber 1 we can treat the information about the state of the other chambers as the coefficients of the terms $a_0 e^{\alpha_0}|0\rangle_1$ and $a_1 e^{\alpha_1}|1\rangle_1$ as

$$\begin{aligned}\Psi_{3f} = a_0 e^{i\alpha_0}|0\rangle_1[b_0 e^{i\beta_0} c_0 e^{i\gamma_0}|0\rangle_2|0\rangle_3 + b_0 e^{i\beta_0} c_1 e^{i\gamma_1}|0\rangle_2|1\rangle_3 \\ + b_1 e^{i\beta_1} c_0 e^{i\gamma_0}|1\rangle_2|0\rangle_3] + a_1 e^{i\alpha_1}|1\rangle_1[b_0 e^{i\beta_0} c_0 e^{i\gamma_0}|0\rangle_2|0\rangle_3].\end{aligned} \tag{27}$$

It is obvious that similar rearrangement is valid for other chambers. Such state is considered as a superposition of conditioned products being non-separable, and by definition this is an entangled state. The state for chamber 1 given in (27) describes what will happen when we perform a measurement or look through the left window on u side or the right window on the w side. The result will be either no light is observed or a light is observed. The former case includes three different potential states before the measurement, i.e. (i) the firefly did not light no matter which chamber it was in, (ii) it lit in chamber 3, (iii) it lit in chamber 2. The process in which one of these is selected corresponds to the collapse of wave function (the firefly's light is assumed not to leak into the next chamber from where it is absent).

3.2 Dynamic Entanglement

As is seen from the firefly model, the states will be dynamically selected or rearranged depending on how we make the measurements, i.e. from which side the observation is done. In order to account for this let us use the Kronecker's δ_{kx},

$$\delta_{kx} = \begin{cases} 1 & \text{for } k = x, \\ 0 & \text{for } k \neq x, \end{cases} \tag{28}$$

giving the dynamical description in place of (25) as

$$\begin{aligned}\Psi_{3fd} = (\delta_{ku} + \delta_{kv} + \delta_{kw}) A\, e^{i\lambda_A}|0\rangle + (\delta_{kv} + \delta_{kw}) B e^{i\lambda_B}|1\rangle \\ + (\delta_{ku} + \delta_{kv}) C e^{i\lambda_C}|2\rangle + (\delta_{kw} + \delta_{ku}) E e^{i\lambda_E}|4\rangle\end{aligned} \tag{29}$$

where $k = u,\ v$, or w (faces of the box). Then, if we focus on each measurement from any one of the three sides, the non-lit state could be eliminated using the Kronecker δ as below. This apparently deprives the firefly of the freedom to stop lighting, resulting in the always lit case

$$\begin{aligned}\Psi_{3fd} = (1 - \delta_{ku})(1 - \delta_{kv})(1 - \delta_{kw}) A\, e^{i\lambda_A}|0\rangle + (\delta_{kv} + \delta_{kw}) B e^{i\lambda_B}|1\rangle \\ + (\delta_{ku} + \delta_{kv}) C e^{i\lambda_C}|2\rangle + (\delta_{kw} + \delta_{ku}) E e^{i\lambda_E}|4\rangle.\end{aligned} \tag{30}$$

Furthermore, it is permissible to assign probability 1 to each side to get the lit state either through the left window or the right window, because the probability can go up to 1. Then the terms undergo renormalization to yield

$$\begin{aligned}\Psi_{3fdn} = \frac{1}{\sqrt{2}}[(1 - \delta_{ku})(1 - \delta_{kv})(1 - \delta_{kw})|0\rangle + (\delta_{kv} + \delta_{kw})|1\rangle. \\ + (\delta_{ku} + \delta_{kv})|2\rangle + (\delta_{kw} + \delta_{ku})|4\rangle],\end{aligned} \tag{31}$$

where all phases (λ's) are assumed to be zero for simplicity without loosing generality. Alternatively, this may also be expressed as

$$\Psi_{3fdn} = \frac{1}{\sqrt{2}} \sum_{k=1,2,4} [(1-\delta_{ij})(1-\delta_{ki})(1-\delta_{kj})\{|i\rangle + |j\rangle\}|\bar{k}\rangle]$$

$$\text{for all } u, v, w \text{ measurements } i, j = 1, 2, 4 \tag{32}$$

$$= \frac{1}{\sqrt{2}}(\delta_{k1} + \delta_{k4} + \delta_{k2})[(1-\delta_{ij})(1-\delta_{ki})(1-\delta_{kj})\{|i\rangle + |j\rangle\}|\bar{k}\rangle]$$

$$\text{for all } u, v, w \text{ measurements } i, j, k = 1, 2, 4 \tag{33}$$

$$= \frac{1}{\sqrt{2}}[\delta_{k1}\{|4\rangle + |2\rangle\}|\bar{1}\rangle + \delta_{k4}\{|2\rangle + |1\rangle\}|\bar{4}\rangle + \delta_{k2}\{|1\rangle + |4\rangle\}|\bar{2}\rangle] \tag{34}$$

where the — over the ket denotes the negation of the bit state. For example, $|1\rangle = |001\rangle$ is negated as $|\bar{1}\rangle = |\overline{001}\rangle$ meaning $|0\rangle = |000\rangle$ or $|2\rangle = |010\rangle$ or $|4\rangle = |100\rangle$, That is, not lit anywhere or absent from the chamber 3. The term as $|4\rangle\ |\bar{1}\rangle$ in (34) reduces to e.g. $|4\rangle$ by the orthogonality among the states.

Then (31) demonstrates the dynamical or operational nature of the experiment. Although the selection of the side from which the measurement will be performed is in the hand of experimentalist, thus is external to the quantum system, that factor is accounted for by the Kronecker δ. For example, if the measurement is done from side u, terms of states $|0\rangle$ and $|1\rangle$ are discarded, leaving only

$$\frac{1}{\sqrt{2}}[(1+0)|2\rangle + (0+1)|4\rangle] = \frac{1}{\sqrt{2}}[|2\rangle + |4\rangle], \tag{35}$$

which is the superposition representing the probability of exactly $\frac{1}{2}$ of getting one of the two states $|2\rangle$ and $|4\rangle$.[3] This discrimination of states $|1\rangle$ may be interpreted as the result of temporary absence of the firefly from chamber 3 at the time of measurement, and alternatively interpreted as the immediate moving out of firefly from chamber 3 into chamber 1 or 2 just at the time of measurement. Possibility of such immediate or instantaneous motion is granted by quantum mechanics, as in the case of the EPR paradox, where a quantum state (ex. an electron spin state) or its information travels instantly through arbitrary distance, exceeding the speed of light as the result of the collapse of the entangled wavefunction [5]. Thus the alternative interpretation above requires that our firefly be a quantum entity, and should not be any real solid matter or creature.

It is obvious that there are entanglements in the state Ψ_{3fd} of (30) or in Ψ_{3fdn} of (31), because states $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|4\rangle$ are product states consisting of products of states of each constituent bit. Which product state is actually involved is determined by the Kronecker δ reflecting the dynamical selection done by experimentalist. So, it is natural to call such situation a *dynamic entanglement*.[4]

[3] If we employ a 4-dimensional state space embedded in the 8-dimensional Hilbert space [3], this state may be indicated as $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which is a vertex of a simplex and represents a dispersive pure state, but different from the other four pure state vertexes on the axes or at the origin that are not dispersive (See [3], Example 6A.16).

[4] Alternatively, it may also be said as dynamic selection of a component in the superposition.

4 Quantum Gates

The entanglement generating function of quantum gates is important and essential to perform quantum super-parallelism in quantum computing. The inverse effect i.e. disentanglement is also important and necessary because we need to focus on the unique or nearly unique solution from the tremendous numbers of superposed intermediate results. So, the effect of the major quantum gates; controlled not (CN), controlled controlled not (CCN), controlled rotation (CR), and controlled phase shift (CPS), are examined with respect to the general separable states given in previous Sects. 2 and 3.

4.1 Controlled Not Gate

Let us denote a CN gate that works on condition of the control bit i over the target bit k as $P_{CNi,k}$. This gate just flips the state of the target bit if the control bit is on, i.e. 1, so if this works on the general separable states of (2) and (16) only the order of the summation will be changed between some consecutive pairs of terms, and the order inside each pair is retained, so that the separability conditions are maintained for both amplitude and phase.

As an example the effect of the CN gate on the 3-bit state Ψ_{123} of (2) is demonstrated below.

$$
\begin{aligned}
P_{CN1,2}\Psi_{123} &= P_{CN1,2}[Ae^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + De^{i\lambda_D}|3\rangle \\
&\quad + Ee^{i\lambda_E}|4\rangle + Fe^{i\lambda_F}|5\rangle + Ge^{i\lambda_G}|6\rangle + He^{i\lambda_H}|7\rangle] \\
&= a_0e^{i\alpha_0}b_0e^{i\beta_0}c_0e^{i\gamma_0}|0\rangle_1|0\rangle_2|0\rangle_3 + a_0e^{i\alpha_0}b_0e^{i\beta_0}c_1e^{i\gamma_1}|0\rangle_1|0\rangle_2|1\rangle_3 \\
&\quad + a_0e^{i\alpha_0}b_1e^{i\beta_1}c_0e^{i\gamma_0}|0\rangle_1|1\rangle_2|0\rangle_3 + a_0e^{i\alpha_0}b_1e^{i\beta_1}c_1e^{i\gamma_1}|0\rangle_1|1\rangle_2|1\rangle_3 \\
&\quad + P_{CN1,2}[a_1e^{i\alpha_1}b_0e^{i\beta_0}c_0e^{i\gamma_0}|1\rangle_1|0\rangle_2|0\rangle_3 + a_1e^{i\alpha_1}b_0e^{i\beta_0}c_1e^{i\gamma_1}|1\rangle_1|0\rangle_2|1\rangle_3 \\
&\quad + a_1e^{i\alpha_1}b_1e^{i\beta_1}c_0e^{i\gamma_0}|1\rangle_1|1\rangle_2|0\rangle_3 + a_1e^{i\alpha_1}b_1e^{i\beta_1}c_1e^{i\gamma_1}|1\rangle_1|1\rangle_2|1\rangle_3] \\
&= A\ e^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + De^{i\lambda_D}|3\rangle \\
&\quad + Ge^{i\lambda_G}|4\rangle + He^{i\lambda_H}|5\rangle + Ee^{i\lambda_E}|6\rangle + Fe^{i\lambda_F}|7\rangle
\end{aligned} \tag{36}
$$

leaving the ratios of the amplitudes $\frac{G}{H} = \frac{E}{F} = \frac{c_0}{c_1}$ and the differences of the phases $\lambda_G - \lambda_H = \lambda_E - \lambda_F = \gamma_0 - \gamma_1$ unchanged, satisfying the separability conditions in (12) and (15).

4.2 Controlled Controlled Not Gate

Let us denote a CCN gate working on condition of the two control bits i and j over the target bit k as $P_{\mathrm{CCN}ij,k}$. This gate flips the state of the target bit if the control bits are simultaneously on, i.e. 11. If this works on the general separable states of (2) and (16), the separability of the resulting state depends on whether the absolute values of the coefficients of the two terms that have 1's simultaneously are equal, and also on the equality of phases of that terms, as seen in the following example.

As an example the effect of the CCN gate on the 3-bit separable state Ψ_{123} of (2) is demonstrated below.

$$\begin{aligned}
P_{\mathrm{CCN}\,12,3}\Psi_{123} &= P_{\mathrm{CCN}\,12,3}[A\,e^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + De^{i\lambda_D}|3\rangle \\
&\quad + Ee^{i\lambda_E}|4\rangle + Fe^{i\lambda_F}|5\rangle + Ge^{i\lambda_G}|6\rangle + He^{i\lambda_H}|7\rangle] \\
&= a_0e^{i\alpha_0}b_0e^{i\beta_0}c_0e^{i\gamma_0}|0\rangle_1|0\rangle_2|0\rangle_3 + a_0e^{i\alpha_0}b_0e^{i\beta_0}c_1e^{i\gamma_1}|0\rangle_1|0\rangle_2|1\rangle_3 \\
&\quad + a_0e^{i\alpha_0}b_1e^{i\beta_1}c_0e^{i\gamma_0}|0\rangle_1|1\rangle_2|0\rangle_3 + a_0e^{i\alpha_0}b_1e^{i\beta_1}c_1e^{i\gamma_1}|0\rangle_1|1\rangle_2|1\rangle_3 \\
&\quad + a_1e^{i\alpha_1}b_0e^{i\beta_0}c_0e^{i\gamma_0}|1\rangle_1|0\rangle_2|0\rangle_3 + a_1e^{i\alpha_1}b_0e^{i\beta_0}c_1e^{i\gamma_1}|1\rangle_1|0\rangle_2|1\rangle_3 \\
&\quad + P_{\mathrm{CCN}\,12,3}[a_1e^{i\alpha_1}b_1e^{i\beta_1}c_0e^{i\gamma_0}|1\rangle_1|1\rangle_2|0\rangle_3 + a_1e^{i\alpha_1}b_1e^{i\beta_1}c_1e^{i\gamma_1}|1\rangle_1|1\rangle_2|1\rangle_3] \\
&= A\,e^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + De^{i\lambda_D}|3\rangle \\
&\quad + Ee^{i\lambda_E}|4\rangle + Fe^{i\lambda_F}|5\rangle + He^{i\lambda_H}|6\rangle + Ge^{i\lambda_G}|7\rangle \qquad (37)
\end{aligned}$$

now, this no longer satisfies the separability conditions all the time, because the last ratio of the amplitudes becomes $\frac{H}{G}$, and the last differences of the phases $\lambda_H - \lambda_G$ in (12) and (15).

In order to maintain the separability throughout the CCN operation, the amplitudes and phases should satisfy the conditions of (12) and (15) as below. For the amplitudes

$$\frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \frac{G}{H} = \frac{H}{G} \quad \text{amounting to } = \pm 1 \qquad (38)$$

and for the phases

$$\lambda_A - \lambda_B = \lambda_C - \lambda_D = \lambda_E - \lambda_F = \lambda_G - \lambda_H = \lambda_H - \lambda_G$$

resulting in

$$\lambda_A = \lambda_B = \lambda_C = \lambda_D = \lambda_E = \lambda_F = \lambda_G = \lambda_H \qquad (39)$$

within trigonometric periodicity.

Similar situations are easily shown by examining other cases $P_{\mathrm{CCN}\,31,2}$ and $P_{\mathrm{CCN}\,23,1}$. Therefore, the 3-bit state that stays separable under the CCN operation is

$$A\,e^{i\lambda_A}[|0\rangle \pm |1\rangle + |2\rangle \pm |3\rangle + |4\rangle \pm |5\rangle + |6\rangle \pm |7\rangle] \qquad (40)$$

where A and λ_A are arbitrary real numbers.

On the other hand, when the conditions of (38) and (39) are not satisfied, the CCN gate creates *entanglements*. As an example, the case of $P_{\mathrm{CCN}\,12,3}$ applying on Ψ_{123} of (2) with $G \neq \pm H$ and $\lambda_G \neq \lambda_H$ amounts to the followings.

$$\begin{aligned}
P_{\mathrm{CCN}\,12,3}\Psi_{123} &= [a_0e^{i\alpha_0}b_0e^{i\beta_0}|0\rangle_1|0\rangle_2 + a_0e^{i\alpha_0}b_1e^{i\beta_1}|0\rangle_1|1\rangle_2 \\
&\quad + a_1e^{i\alpha_0}b_0e^{i\beta_1}|1\rangle_1|0\rangle_2][c_0e^{i\gamma_0}|0\rangle_3 + c_1e^{i\gamma_1}|1\rangle_3] \\
&\quad + a_1e^{i\alpha_0}b_1e^{i\beta_1}|1\rangle_1|1\rangle_2[c_1e^{i\gamma_1}|0\rangle_3 + c_0e^{i\gamma_0}|1\rangle_3]. \qquad (41)
\end{aligned}$$

This clearly shows that the factorization of the bit-3 can not make further progress unless $c_0 = c_1$ and $\gamma_0 = \gamma_1$ or $c_0 = -c_1$ and $\gamma_0 = \gamma_1 \pm (2n-1)\pi$ for any integer n. This corresponds to the former condition $G = \pm H$ in (38), and $\lambda_G = \lambda_H + n\pi$ in (39). However, these are prohibited as $G \neq \pm H$ and $\lambda_G \neq \lambda_H$, so it is not perfectly or totally separable.

The effect of CCN gate $P_{\mathrm{CCN}\,ij,k}$ on the n-bit separable state $\Psi_{12\cdots n}$ can be examined by applying it to (16), getting similar results.

4.3 Controlled Rotation Gate

4.3.1 SU(2) Representation

Next we denote a CR gate working on condition of the control bit i over the target bit k as operators $R^{\theta}_{xi,k}$, $R^{\theta}_{yi,k}$, and $R^{\phi}_{zi,k}$, where x, y, and z specifies the axes of rotation and θ and ϕ are the angles of rotation. This operator is designed to operate on the state represented as an two dimensional vector, adopting the standard notation in SU(2) (special unitary group) [6]. In this system, a three dimensional unit vector having the polar coordinates $r = 1$, θ, and ϕ, or the Cartesian counterparts $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, and $z = \cos\theta$, is expressed in two dimensional vector form as

$$\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}, \tag{42}$$

corresponding to our bit as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{with } \theta = 0,\ \phi = 0, \tag{43}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{with } \theta = \frac{\pi}{2},\ \phi = 0, \tag{44}$$

being the usual convention in quantum computing community. If we put a Cartesian frame, the bit state $|0\rangle$ is a unit vector pointing up from the origin along the z axis, and the bit state $|1\rangle$ is a unit vector along the x axis. The y axis is for the imaginary part coming from the complex number in the quantum mechanical probability amplitude. In other words this is a method to represent the quantum bit state as a spinor.[5]

Now, the operators of the three major rotations $R^{\theta}_{xi,k}$, $R^{\theta}_{yi,k}$, and $R^{\phi}_{zi,k}$ are formulated in this SU(2) system,

$$R^{\theta}_{xi,k} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{45}$$

$$R^{\theta}_{yi,k} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{46}$$

$$R^{\phi}_{zi,k} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} \tag{47}$$

waiting to operate on the state vectors such as given in (43) and (44). The effect of these operations on our states $|0\rangle$, $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$, $|5\rangle$, $|6\rangle$, and $|7\rangle$ are easily examined, as shown in the following section.

[5] In this method, rotation by an angle θ around x axis and y axis in the formula (R^{θ}) only rotate the vector by half angle $\frac{\theta}{2}$ in the state space, so that the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are perpendicular to each other illustrating the orthogonality.

4.3.2 Controlled Rotation Around y Axis

The results of controlled rotation around y axis is demonstrated. First of all, the operation on the states $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|3\rangle$ bears no change because in these cases the control bit is 0.

$$R^{\theta}_{y1,2}|0\rangle = |0\rangle, \qquad R^{\theta}_{y1,2}|1\rangle = |1\rangle, \qquad R^{\theta}_{y1,2}|2\rangle = |2\rangle, \qquad R^{\theta}_{y1,2}|3\rangle = |3\rangle. \tag{48}$$

Then, as an example when the operation cause real change, the case of rotation around y axis $R^{\theta}_{y1,2}$ on the state $|4\rangle$ is detailed below.

$$\begin{aligned} R^{\theta}_{y1,2}|4\rangle &= R^{\theta}_{y1,2}|1\rangle_1|0\rangle_2|0\rangle_3 = \begin{pmatrix}0\\1\end{pmatrix}_1 R^{\theta}_{y1,2}\begin{pmatrix}1\\0\end{pmatrix}_2\begin{pmatrix}1\\0\end{pmatrix}_3 \\ &= \begin{pmatrix}0\\1\end{pmatrix}_1\left[\cos\frac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix}_2 + \sin\frac{\theta}{2}\begin{pmatrix}0\\1\end{pmatrix}_2\right]\begin{pmatrix}1\\0\end{pmatrix}_3 \\ &= \cos\frac{\theta}{2}|1\rangle_1|0\rangle_2|0\rangle_3 + \sin\frac{\theta}{2}|1\rangle_1|1\rangle_2|0\rangle_3 \\ &= \cos\frac{\theta}{2}|4\rangle + \sin\frac{\theta}{2}|6\rangle \end{aligned} \tag{49}$$

similarly results of operation on states $|5\rangle$, $|6\rangle$, and $|7\rangle$ are derived.

Next, the operation of the rotation around the y axis $R^{\theta}_{y1,2}$ on our 3-bit separable state Ψ_{123} given by (2) is derived.

$$\begin{aligned} R^{\theta}_{y1,2}\Psi_{123} = R^{\theta}_{y1,2}[&A\, e^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + De^{i\lambda_D}|3\rangle \\ &+ Ee^{i\lambda_E}|4\rangle + Fe^{i\lambda_F}|5\rangle + Ge^{i\lambda_G}|6\rangle + He^{i\lambda_H}|7\rangle] \end{aligned} \tag{50}$$

$$\begin{aligned} &= A\, e^{i\lambda_A}|0\rangle + Be^{i\lambda_B}|1\rangle + Ce^{i\lambda_C}|2\rangle + De^{i\lambda_D}|3\rangle \\ &\quad + \cos\frac{\theta}{2}Ee^{i\lambda_E}|4\rangle + \left[\cos\frac{\theta}{2}Fe^{i\lambda_F} - \sin\frac{\theta}{2}Ge^{i\lambda_G} - \sin\frac{\theta}{2}He^{i\lambda_H}\right]|5\rangle \\ &\quad + \left[\sin\frac{\theta}{2}Ee^{i\lambda_E} + \cos\frac{\theta}{2}Ge^{i\lambda_G}\right]|6\rangle + \left[\sin\frac{\theta}{2}Fe^{i\lambda_F} + \cos\frac{\theta}{2}He^{i\lambda_H}\right]|7\rangle. \end{aligned} \tag{51}$$

Then the separability conditions are examined. Amplitude and phase conditions are same from A to D or λ_A to λ_D, but the rest have been significantly changed.

Applying the result of (51) to the separability conditions in (12).

$$\text{[before Rotation;]}\frac{A}{B} = \cdots = \frac{E}{F} = \frac{G}{H}, \tag{52}$$

[after Rotation;]

$$\begin{aligned} \frac{E'}{F'} &= \frac{|\cos\frac{\theta}{2}Ee^{i\lambda_E}|}{|\cos\frac{\theta}{2}Fe^{i\lambda_F} - \sin\frac{\theta}{2}Ge^{i\lambda_G} - \sin\frac{\theta}{2}He^{i\lambda_H}|} \\ &= \frac{\cos\frac{\theta}{2}E}{\sqrt{2[-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos(\lambda_F-\lambda_G)FG+\sin^2\frac{\theta}{2}\cos(\lambda_G-\lambda_H)GH-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos(\lambda_F-\lambda_H)FH]+\cos^2\frac{\theta}{2}F^2+\sin^2\frac{\theta}{2}[G^2+H^2]}} \end{aligned} \tag{53}$$

for the amplitude of the fifth and sixth terms. As it looks, the result of rotation given by (53) is different from the original value of (52) in general,[6] violating the separability condition at least unless $\sin\frac{\theta}{2}=0$, i.e. $\theta=0,\pm 2n\pi$, and the rotation reduce to

$$R^{2n\pi}_{y1,2}=\begin{pmatrix}\cos(\pm\pi) & -\sin(\pm\pi)\\ \sin(\pm\pi) & \cos(\pm\pi)\end{pmatrix}=\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\pm\mathbf{I},$$

which is an identity or just a flipping of the vector direction. Otherwise, the violation means that the rotation did the *entanglement*. Similarly the ratio of G and H also shows change.

$$\text{[before Rotation;]}\quad \frac{A}{B}=\cdots=\frac{E}{F}=\frac{G}{H},\tag{54}$$

$$\text{[after Rotation;]}\quad \frac{G'}{H'}=\frac{|\sin\frac{\theta}{2}Ee^{i\lambda_E}+\cos\frac{\theta}{2}Ge^{i\lambda_G}|}{|\sin\frac{\theta}{2}Fe^{i\lambda_F}+\cos\frac{\theta}{2}He^{i\lambda_H}|}.\tag{55}$$

This violate the separability condition again unless $\sin\frac{\theta}{2}=0$, i.e. $\theta=0,\pm 2n\pi$.

4.3.3 Controlled Rotation Around x and z Axes

Controlled Rotation Around x Axis The results of

$$R^{\theta}_{xi,k}=\begin{pmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$$

given in (45) is illustrated by an example with $R^{\theta}_{x1,2}$ acting on a 3-bit state $|4\rangle$ in (2).

$$R^{\theta}_{x1,2}|4\rangle=\cos\frac{\theta}{2}|4\rangle-i\sin\frac{\theta}{2}|6\rangle.\tag{56}$$

Controlled Rotation Around z Axis The rotation around z axis on the $|4\rangle$ state; $R^{\phi}_{zi,k}\,|4\rangle=e^{-i\frac{\phi}{2}}\,|4\rangle$, amounts to just a phase factor $e^{-i\frac{\phi}{2}}$. Likewise, the other cases are easily examined.

4.3.4 Controlled Controlled Rotation Around y Axis

Similar to the CCN gate, we have CCR gates $R^{\theta}_{xij,k}$, $R^{\theta}_{yij,k}$, and $R^{\phi}_{zij,k}$, rotating the target bit k on condition of the two control bits i and j. As an example let us look at the results of y rotation $R^{\theta}_{y12,3}$ on the state $|6\rangle$.

$$R^{\theta}_{y12,3}|6\rangle=\cos\frac{\theta}{2}|6\rangle+\sin\frac{\theta}{2}|7\rangle.\tag{57}$$

These states $|6\rangle$ and $|7\rangle$ are the only terms in the general 3-bit separable state Ψ_{123} given in (2), that undergo changes by the CCR operation $R^{\theta}_{y12,3}$.

[6]The denominator of (53) should be synthesized into a single amplitude and a phase factor even it may be fairly complicated. Then we can talk the separability in more definite terms, and find out some exceptionally separable case, if there is any.

4.4 Controlled Phase Shift Gate

As is already shown in Sect. 4.3.3 the rotation around z axis $R^{\phi}_{zi,k}$ works as a controlled phase shift when applied to the $|0 \text{ or } 1\rangle_i$ and $|0 \text{ or } 1\rangle_k$ bits. However, it is convenient to think of a simple phase shifting operation $S^{\xi}_{i,k}$ such as below.

$$S^{\xi}_{i,k}\Psi_{123} = e^{i\xi}\Psi_{123} \quad \text{with } k = 1, 2, 3;\, i \neq k. \tag{58}$$

5 Concluding Remarks

We have developed a theoretical framework to describe degrees of entanglement of bit states in quantum computing. We derived amplitude criteria ((13) and (19)) and phase criteria ((15) and (20)) characterizing entangled unit vector states in Hilbert space.

Using these criteria, we looked at the possibility of different quantum gates, such as controlled not (CN), controlled controlled not (CCN), controlled rotation (CR), and controlled phase shift (CPS), to create the entanglements. Furthermore, the selection of measurement mode external to the quantum system is incorporated in the formula using Kronecker delta (δ_{kx}), introducing the concept of dynamic entanglement. With this the process of wavefunction collapse upon the measurement from outside, is understood as the result of the activation of the dynamic entanglement. A superposed pure state having intrinsic (ontological) uncertainty in a parametric state space, is analyzed to be a dynamically entangled state in Hilbert space.

Acknowledgements The authors are grateful to Leanne Robertson at Smith College for her kind discussions in solving simultaneous equations deriving the criteria.

References

1. Matsueda, H., Cohen, D.W.: Reduction of theoretical uncertainty in quantum computing. Int. J. Theor. Phys. **38**(2), 701–711 (1999)
2. Matsueda, H.: Solid state quantum computation. In: Peřina, J. (ed.) Coherence and Statistics of Photons and Atoms, pp. 422–469. Wiley, New York (2001)
3. Cohen, D.W.: An Introduction to Hilbert Space and Quantum Logic. Springer, Berlin (1989). Chaps. 1, 3, and 6
4. Brassard, G., Mor, T.: Multi-particle entanglement via two-particle entanglement. In: Williams, C.P. (ed.) QCQC'98. Lecture Notes in Computer Science, vol. 1509, pp. 1–9. Springer, Berlin (1999)
5. Einstein, A., Podolsky, B., Rosen, N.: Phys. Rev. **47**, 777–780 (1935)
6. Mathews, J., Walker, R.L.: Rotation operators and Euler angles. In: Mathematical Methods of Physics, 2nd edn., pp. 403–405. Benjamin, Reading (1970)